Fractals and Music

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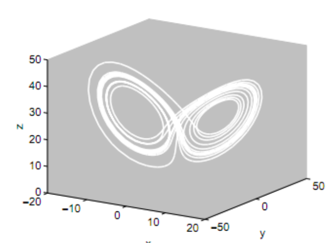
Introduction

Intent

When looking for ideas for a project, I was drawn to the study of different patterns and their applications. The beauty of fractals intrigued me from the start. The combination of simplicity and complexity creates something completely unique. What is especially interesting about them is that they all stem from mathematical formulas. If mathematical formulas can create aesthetically pleasing fractal art, then why wouldn't they be able to create aesthetically pleasing fractal music? This is what I sought to find out.

Chaos Theory

Before we begin discussing what fractals are, we will first discuss a little bit about where they came from. Fractals first stemmed out of the study of chaos theory. Chaos theory is the study of dynamical systems that are largely dependent on their initial conditions, that is, if one would change the initial condition, the whole system would be completely different. This has been known in many other disciplines as the *butterfly effect*, named after one chaotic system that looks like a butterfly (see figure), and made famous in 1972 with Edward Lorenz' book, "Does the Flap of a Butterfly's Wings in Brazil Set Off a Tornado in Texas" [**1**].

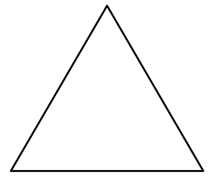


The word *chaos* in chaos theory is a bit misleading. What is happening in the system is not actually random, though it may look random. In actuality, there is a complex pattern that creates something called a *strange attractor*, and it attracts the solutions to a range of particular solutions. The butterfly in the one particular chaotic system was the strange attractor of that solution. This attractor is called the Lorenz Attractor. When these attractors are visualized, they are often called fractals.

Fractals

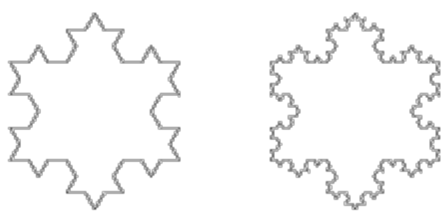
Fractals are traditionally thought of as a visual pattern having the characteristic of self-similarity. They are generated by repeating a specified rule to a desired number of iterations. In 1977, Benoit B. Mandelbrot coined the term "fractal" and studied much of its geometry, specifically how fractals are able to mimic nature in a curious way [**2**]. Fractals can be generated to look like mountains, trees, and snowflakes, among other natural phenomena. This is one of the interesting things about chaos theory. Something that may look random could have an underlying pattern. This means we can replicate the pattern to make something look random, such as the peaks and valleys of a mountain range. This is something that is often used in computer graphics for popular movies such as a planet in Star Trek 2: The Wrath of Khan, and splashing lava in Star Wars: Episode 3 [**3**].

We will first use the example of the "snowflake curve," also known as the triadic von Koch curve [**4**]. First we start with a simple equilateral triangle:

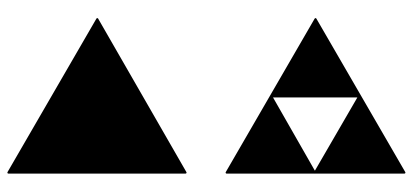


The rule we will use for this triangle is that for each edge, we will divide it into 3 equal parts. With the middle part, we will form another equilateral triangle.

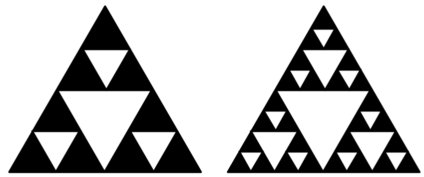
This is one iteration. Doing this to many iterations, the form of a snowflake begins to take shape. This is our first example of a fractal. Each edge looks like every other edge, as seen from the images from [**5**].



There is not a specific rule that says we have to make a fractal in this way. Another way we can make a fractal is by changing the inside of the shape. Let us start again with an equilateral triangle. This time we will inscribe an upside-down triangle.



This is one iteration. Our next iteration, we will do the same thing to every right-side-up triangle. We could hypothetically do this an infinite number of times.



What we have made is called Sierpinski's Triangle, first discovered in 1915 [**6**]. If we look at one of the inner right-side-up triangles, it looks exactly like the large whole triangle. This quality of being able to zoom in on figures, or zoom out, and have the picture look exactly the same is intriguing about fractals.

In nature, this is seen with how the branching of one tree limb is similar to the branching of the whole tree [**3**]. In a similar way, if we break off one piece of broccoli from a stalk, the piece looks like the whole stalk. This is the essence of self-similarity.

Music

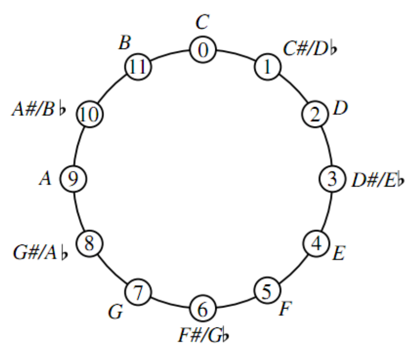
For those not familiar with music theory, we will cover the basics. The typical scale includes twelve notes: C, C#, D, D#, E, F, F#, G, G#, A, A#, and B. Different combinations of these notes make up different key signatures. For instance, we will often be working with the C key signature, which orders the notes as such: C, D, E, F, G, A, and B. These notes played to different melodies often sound the best together. Note that this only includes seven of the twelve notes. We will sometimes be using only these seven, and other times we will use all twelve.

Since this study will be including patterns of numbers, we will often use numbers instead of letters for the notes. Mathematically, we will make the numbers modulo 7 (mod 7) when using the C key signature, or mod 12 when using all of the notes. Using mod 7 as an example, we have the numbers 0 through 6. The number from 0 to 6 that we choose will be the number that remains after we have subtracted off 7 enough times. For example, if the number 7 appears, 7 - 7 = 0, and so we will use 0. If the number 8 appears, 8 – 7 = 1, and so we will use 1. If the number 23 appears, we will subtract 7 three times until we get to 2, and so we will use 2. Another way to write this is 23 mod 7 $\equiv$ 2. As noted before, each of these numbers will be assigned to a note.

When thinking about mod 12, is it easy to think of it as a clock. If we begin at 1 o'clock, and add 13 hours, we are now at 2 o'clock [**7**]. In the same way,

$$1 + 13 = 14$$
$$14 \bmod 12 \equiv 2$$

From this idea, we have the musical clock described in [8].



Looking Ahead

In this paper, we will first discuss how we can apply integer sequences with fractal properties to create melodies. We will then look at how these melodies contain transformations that are similar to those that exist in music already. Looking at music that already exists, we will explore how music naturally contains fractal properties. Finally, we will evaluate the results and analyze the implications.

Fractal Integer Patterns

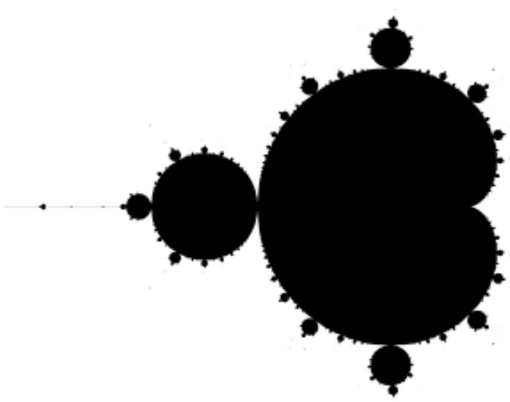
Mandelbrot Equation

We created fractals by taking simple shapes and using a rule to develop the fractals from the shapes. We can create fractal integer sequences by choosing a number, applying a rule to it, and resulting in a string of numbers. Mandelbrot, the man who coined the name fractal, did this through his equation:

$$Z \leftrightarrow z^2 + c.$$

We use this equation by starting with an initial z-value, and a fixed c-value. After calculating the right side, this calculated value becomes our new z-value, and it is fed back into the equation [**9**]. By letting the first $z = c$, Mandelbrot would plot the c values onto the complex plane which yielded a closed orbit [**10**]. For example, if we start with $c = z = i$, where $i = \sqrt{-1}$ then the following z's will be

$$z = i^2 + i = -1 + i$$
$$z = (-1+i)^2 + i = -i$$
$$z = (-i)^2 + i = -1 + i,$$



and since $-1 + i$ is already in the orbit, then the orbit is closed and i is plotted on the complex plane. This yields the fractal called the Mandelbrot set pictured on the left [**11**]. The self-similarity can be seen along edges.

The string of complex numbers can give us sets of coordinates of the complex plane, with first coordinate as the real portion and the second coordinate with the imaginary portion. To get a single string of integers, we are able to choose real integers for our initial values with the underlying equation giving us self-similarity. For example, if we start with $z = 0$ and $c = 1$, the sequence will be 0, 1, 2, 5, 26, etc. The possibilities for different sequences are endless given any self-fed formula.

Morse-Thue Sequence

The Morse-Thue sequence is also called a "ones-counting sequence." It is called this since it is generated by listing numbers in their binary notation, and counting how many ones are in each number. Here are the binary numbers from zero to ten:

0, 1, 10, 11, 100, 101, 110, 111, 1000, 1001, 1010.

Now we will count how many ones are in each number, and list them again:

0, 1, 1, 2, 1, 2, 2, 3, 1, 2, 2.

This integer sequence just by appearance does not seem to have fractal characteristics, until we examine it further. Since a string of numbers cannot be literally magnified, then we can define magnify in another way. One way to simulate magnification is to take away every other number. If we were to take away every other number of this sequence, beginning with the first 1, we have

0, ~~1~~, 1, ~~2~~, 1, ~~2~~, 2, ~~3~~, 1, ~~2~~, 2

0, 1, 1, 2, 1, 2.

This is the same sequence that we originally started with. Thus, we have self-similarity in the sequence, which is what makes it a fractal integer sequence [**9**].

Not only can we “zoom in” on the sequence, we can also “zoom out.” If we replace every number in the sequence with itself and 1 plus itself, then we come up with the same sequence. So, 0, 1, 1, 2 becomes

0, 0+1, 1, 1+1, 1, 1+1, 2, 2+1

0, 1, 1, 2, 1, 2, 2, 3.

This adds to the self-similarity.

We can also generate the sequence using a certain rule, as we did with the equilateral triangles. This time, we start with a 0. The rule we will use is that for the next 2^m terms, we will take the previous 2^m terms and add one to every number [**12**]. Thus, since the first term is 0, the second term is 1. The next two terms will be 0+1 and 1+1 which is 1 and 2. The next four terms will be 0+1, 1+1, 1+1, and 2+1, which is 1, 2, 2, 3. Thus, the sequence is

0, 1, 1, 2, 1, 2, 2, 3.

This sequence yields another famous sequence, known as the “Dress sequence,” after German mathematician Andreas Dress [**12**]. If we take each of these numbers and use them as exponents of 2 (i.e. 2^0, 2^1, 2^1, 2^2, 2^1, 2^2, 2^2, 2^3), we obtain the sequence:

1, 2, 2, 4, 2, 4, 4, 8.

This sequence has similar properties as the Morse-Thue sequence. Generalizing, applying any one-to-one function to this sequence will give us another sequence with fractal qualities.

There are other integer sequences that have these properties that at first glance have nothing to do with the Morse-Thue Sequence, many of which are listed on [**13**].

One example of this is the sequence which lists the number of ways each integer can be written as the sum of squares when order does not matter. For example,

$$0 = 0^2 + 0^2$$

$$1 = 0^2 + 1^2$$

$$2 = 1^2 + 1^2,$$

and so the first three integers of this sequence are 1, 1, 1. Since 3 cannot be written as a sum of squares, then the next integer is 0. Skipping ahead, 25 can be written as $5^2 + 0^2$ and $3^2 + 4^2$. Therefore, the 26th entry of this sequence is 2. This happens to be the first integer which can be written as two different sums of squares. Looking at the first 20 integers, we can see how this sequence contains the same sort of self-similarity:

1, ~~1~~, 1, ~~0~~, 1, ~~1~~, 0, ~~0~~, 1, ~~1~~, 1, ~~0~~, 0, ~~1~~, 0, ~~0~~, 1, ~~1~~, 1, ~~0~~

1, 1, 1, 0, 1, 1, 0, 0, 1, 1.

This sequence does not look like the Morse-Thue sequence and is generated in a very different way, yet it contains the same property of removing every other integer to get the same sequence.

<u>Other Bases</u>

We initially created the Morse-Thue sequence by listing integers in their binary notation, that is, base 2. A question that naturally arises is if this will work in other bases. Since there are more than 1s and 0s in other bases, we can look at base 2 again and see that counting the ones is also adding the integers together. Therefore, we will add the integers together in other bases. We will begin with base 3. These are the integers from 0 to 20:

0, 1, 2, 10, 11, 12, 20, 21, 22, 100, 101, 102, 110, 111, 112, 120, 121, 122, 200, 201, 202.

If we add the integers in each number together, we get the sequence:

0, 1, 2, 1, 2, 3, 2, 3, 4, 1, 2, 3, 2, 3, 4, 3, 4, 5, 2, 3, 4.

Initially, it does not seem that this sequence has the same quality since if we take away every other number, it does not give us the same sequence. Upon further investigation, if we choose every third number, beginning with 0, then we will have the same sequence:

0, 1, 2, **1**, 2, 3, **2**, 3, 4, **1**, 2, 3, **2**, 3, 4, **3**, 4, 5, **2**, 3, 4

0, 1, 2, 1, 2, 3, 2.

We were able to construct the Morse-Thue sequence by starting with a 0 and applying some rule to that 0. Similarly, we are able to create this sequence beginning with a 0. This time, taking our sequence of 3^m terms, the next 3^m terms will be created by adding one to every term, and then the next 3^m terms will be created by adding two to every term. Thus, we create $2*3^m$ terms from our original 3^m. Therefore, our next two terms of our sequence will be 1 and 2. At this point we have 0, 1, and 2. The next 6 terms will be 1, 2, 3, 2, 3, and 4. Thus, we have 0, 1, 2, 1, 2, 3, 2, 3, 4.

Hypothesizing, if we list integers in base *n*, add the integers within each integer to get our sequence, choose every *n*th integer of that sequence, we will arrive at the same sequence. A simple proof of this can be seen that in any base, when we multiply an integer by *n*, then we are simply adding a 0 to the end. When we take every *n*th integer, then we are taking 0*n*, *1n*, 2*n*, 3*n*, 4*n*, and so on. Therefore, for any integer *kn,* *kn* = *k* + 0 = *k*. Thus,

$$0n, 1n, 2n, 3n, 4n, \ldots = 0, 1, 2, 3, 4, \ldots$$

If we create two different sequences by counting the ones for the first sequence and counting the twos for the second sequence, we have the following:

0, 1, 0, 1, 2, 1, 0, 1, 0, 1, 2, 1, 2, 3, 2, 1, 2, 1, 0, 1, 0

0, 0, 1, 0, 0, 1, 1, 1, 2, 0, 0, 1, 0, 0, 1, 1, 1, 2, 1, 1, 2.

By observation, we can see that if we choose every third integer for each sequence, we will get the same two sequences back for the same reason as above. Since multiplying by 3 in base 3 will add a zero to the number, then every third number is 0, 3, 6, 9, ... which will have the same one and two counts as 0, 1, 2, 3, … Similarly, this will work with any base. Therefore, whether we are adding up all of the integers in any base *n*, or adding how many *k*'s are in each integer for some $0 < k < n$, the resulting sequence with have the fractal property of choosing every *n*th integer and having the result of the same sequence.

Interestingly, the two sequences we created by counting the ones and twos of integers base 3 are a linear combination of our sequence that we created by adding up all of the integers within each integer. If we let our ones-counting sequence be A_1, and our twos-counting sequence be A_2, then our first sequence is A_1+2A_2. Similarly, with any base *n*, with *n*-counting sequences A_n, our integer counting sequence will be equal to

$$\sum_{n=1}^{n} nA_n .$$

Integers to Music

Direct Correlation

One of the simplest ways to covert integer sequences to music is convert the numbers to a particular modulus and assign notes to numbers. We will start with the Morse-Thue sequence as an example. We choose the key of C, and assign

C = 0, D = 1, E = 2, F = 3, G = 4, A = 5, B = 6.

If we choose each note to be a quarter note, and we want four measures of music, then we will take the first sixteen numbers of the sequence modulus 7. This gives the following piece:

The beauty of making this piece directly from the fractal integer sequence is that the fractal properties will hold for the string of notes. As with the integer sequence, if we remove every other note from the melody, then the melody remains the same. Also, if we replace each note with itself and one note higher, then the melody will remain the same.

The problem with the piece is that there is no sense of rhythm. To create rhythm, we can make a rule that for every repeated note, we can combine it to one note that is held out longer. For instance, two quarter notes will become one half note. Using this rule, this changes the piece to:

This makes the piece a bit more interesting, even though we lose the property of removing every other note to get the same melody.

We need not restrain ourselves with going in order from C = 0 to B= 6. Using any of our integer sequences, we can assign each number to any note. For instance, if we let

C = 0, D = 3, E = 5, F = 1, G = 4, A = 2, B = 6

then we have the following melody:

Using this technique will give us intervals that are more interesting and perhaps more pleasing to listen to. If we choose our notes carefully, then we can plan how we want our intervals to sound and where we want the piece to go, and even more so when we allow ourselves more notes to choose. To make things even more interesting, we could use our two base 3 integer sequences together, and choose a different set of notes for each sequence.

Integers as Notes and Rhythm

Another simple and interesting way to create music through integers is to assign them not only to notes, but also to rhythm. An example of this would be if we looked at the sets of coordinates given by Mandelbrot's equation. We can assign the first integer to a note, and the second integer to duration. To put some constraints on this, we will take the first integer mod 7, and the second integer mod 4. We choose mod 4 for the duration since we are working with a 4/4 tempo. Each set of coordinates can be assigned to a measure, and if the note does not last the whole measure, the remaining beats of the measure will be used as a rest.

For example, if we choose $z = c = 3i$, our first four coordinates using the real part as the first and the imaginary part as the second yield (0,3), (5,3), (2,1), and (3,3). This gives us the following line of music.

This line not only gives us an interesting melody line, but it also gives us a unique rhythm. There are many ways to play around with this technique. For instance, we could have used a different modular system. Instead of staying within the C key signature, we could have made the first integer mod 12, using all twelve notes. We could have also given it a 3/4 tempo and used mod 3 for the second integer.

Fractal Time Scales

We can also play around with how we scale our integer sequences. For instance, since one of the ideas behind fractals is that it looks similar at every scale, then if we play the integer sequence at several scales at one time, then the whole of the music will contain that fractal quality. We will go back to our first Morse-Thue sequence melody. If we play this at three different scales, this is what it would look like:

When these lines are played together, it sounds as if one is playing the piano with the sustained pedal down. Since every other note gives us the same melody, when the melody is stretched out, the notes will match up to the original melody. This is another way to visualize the fractal qualities of this sequence.

Musical Transformations

Transformations

In our discussion of musical transformations, we will be using all twelve notes. Certain interactions between these notes can be classified through two transformations. When a chord moves from a major to a minor, or moves to a different chord altogether, then it is undergoing a transformation or a composition of transformations. Mathematically, the transformations are performing some kind of addition or subtraction to the pitch represented by a number, or the chord represented by a set of numbers. Note that this discussion of mathematical musical transformations should be distinguished from the musical understanding of transformations. There will be a slight difference in the understanding of transpositions and inversions in music.

Transposition

The first transformation we will consider is the transformation of transposition. When a song is transposed from one key signature to another, all of the notes are undergoing a particular transposition. Each transposition has a number that corresponds with it. We may have a transposition of 2, denoted T_2. In this case, each note has 2 added to it, and then it is taken mod 12. A mathematical notation of this is described in [**9**] as the following:

$$T_n : Z_{12} \to Z_{12}$$
$$T_n(x) := x + n \bmod 12$$

where Z represents the integers. This is not only useful in transposing a whole song, but it is also useful when going from one chord to another. The chord C-major is represented with numbers as 0, 4, and 7. When we transpose this chord by 2, we have $T_2(0, 4, 7) = (2, 6, 9)$, which gives us D-major. In fact, transposing a C-major will always give us another major chord. We will need some other tool to get a minor chord, namely the inversion transformation.

Inversion

Inverting a chord is akin to taking the inverse of a chord. Like a transposition, a number is associated with each inversion. Inverting a note by 2 would be to take the opposite of that note and then add two. The opposite of a note x is simply $-x$ mod 12. Mathematically, this is denoted:

$$I_n : Z_{12} \to Z_{12}$$
$$I_n(x) := -x + n \bmod 12$$

When we apply the inversion of 2 to C-major, we get $I_2(0, 4, 7) = (2, 10, 7)$, which is a G-minor.

Composition of Transformations

The following table found in [**9**] can give us a broader look at the chord structures. To get to a chord in the same column we need to perform a transposition, and to get to the other column we need to perform an inversion.

Major Triads	Minor Triads
$C = \langle 0, 4, 7 \rangle$	$\langle 0, 8, 5 \rangle = f$
$C\sharp = D\flat = \langle 1, 5, 8 \rangle$	$\langle 1, 9, 6 \rangle = f\sharp = g\flat$
$D = \langle 2, 6, 9 \rangle$	$\langle 2, 10, 7 \rangle = g$
$D\sharp = E\flat = \langle 3, 7, 10 \rangle$	$\langle 3, 11, 8 \rangle = g\sharp = a\flat$
$E = \langle 4, 8, 11 \rangle$	$\langle 4, 0, 9 \rangle = a$
$F = \langle 5, 9, 0 \rangle$	$\langle 5, 1, 10 \rangle = a\sharp = b\flat$
$F\sharp = G\flat = \langle 6, 10, 1 \rangle$	$\langle 6, 2, 11 \rangle = b$
$G = \langle 7, 11, 2 \rangle$	$\langle 7, 3, 0 \rangle = c$
$G\sharp = A\flat = \langle 8, 0, 3 \rangle$	$\langle 8, 4, 1 \rangle = c\sharp = d\flat$
$A = \langle 9, 1, 4 \rangle$	$\langle 9, 5, 2 \rangle = d$
$A\sharp = B\flat = \langle 10, 2, 5 \rangle$	$\langle 10, 6, 3 \rangle = d\sharp = e\flat$
$B = \langle 11, 3, 6 \rangle$	$\langle 11, 7, 4 \rangle = e$

If we go back to our example of $I_2(0, 4, 7) = (2, 10, 7)$, we can see that this can also be expressed as an inversion and then a transposition of 2. This can be denoted as $I_2 = T_2I$. This will give us consistency when talking about composition of transformations since there are many ways to get from one chord to another, as it is described in [**8**]. If we wish to only invert the chord, then we can describe this as an inversion and a transposition of 0, denoted T_0I. If we wish to only transpose, then we will leave off the I as we did previously.

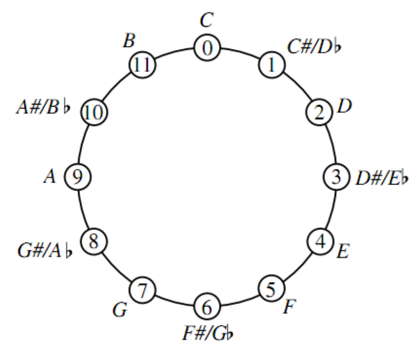
We can also view this set of transformations as a group. The first property of a group is that it is closed. From the previous paragraph, we can see that the composition of any two transformations will give us another transformation. The second property that it must contain is the existence of an identity. In this group, T_0 would bring us back to the same chord, and is thus the identity. The third property it must contain is the existence of an inverse for every element. If we consider the transformation $T_nI(x)$, the transformation done to it is an inversion, $-x$, and then an addition of n. Thus, we have $T_nI(x) = -x + n$. To bring $-x + n$ to the original x, we will first invert it, $x - n$, and then add n

back to it. If the transformation does not contain an inversion, then it is of the form $T_n(x) = x + n$. To bring it back to x, we add a $-n$. Therefore, we have

$$T_n I \circ T_n I = T_0$$
$$T_n \circ T_{-n} = T_0$$

This means that every element has an inverse, and thus it is a group.

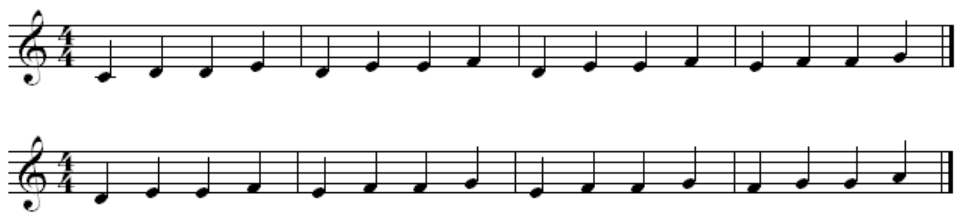
Recall that we can view the 12 notes as a musical clock. These transformations that we have discussed also perform transformations to the 12-gon clock. Performing a transposition on this clock will rotate the 12-gon counterclockwise. Performing an inversion will flip the 12-gon. Therefore, we not only have a group, but we have a group that is isomorphic to the dihedral group, which is the group of symmetries of the polygon.



<u>Transformations in Music</u>

These transformations will often be found in music in moving from line to line or chord to chord. Joseph Straus in *Introduction to Post-Tonal Theory* uses these transformations when analyzing pieces of music, revealing where and why they are used. In analyzing Schoenberg's *Book of the Hanging Gardens*, he states that certain lines are "transposed and ordered so as to reproduce, over a large span, the intervallic succession of the opening melodic gesture" [**7**]. In essence, it is a tool that allows us to recall a familiar part of the song in an interesting way.

These transformations can also be found in our fractal generated music. We will again look at our first melody that we created through the Morse-Thue Sequence, except this time we will extend it to eight measures:



We can see that each measure looks similar to every other measure. Since we were working with mod 7, we will continue to work in mod 7 for these transformations. The second measure is T_1 of the first measure, since

$$T_1(0, 1, 1, 2) = (1, 2, 2, 3).$$

Similarly, the fourth measure is T_2 of the first measure, or T_1 of the second and third measures, since

$$T_2(0, 1, 1, 2) = T_1(1, 2, 2, 3) = (2, 3, 3, 4).$$

If we let the first measure be $T_0(0, 1, 1, 2)$, and we list out the measures in terms of this measure, we have

$$T_0, T_1, T_1, T_2, T_1, T_2, T_2, T_3.$$

Taking out the numbers, we have

0, 1, 1, 2, 1, 2, 2, 3.

This is the beginning of the Morse-Thue sequence. Thus, if we begin with a measure of our choosing, and then perform these same transformations on that measure to generate the following measures, then we will have a piece that is fractal.

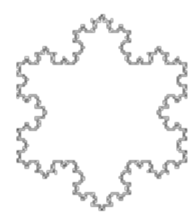
We can conclude from this discussion of transformations that our fractal generated music does have similar properties of musician composed music. This should not be surprising since the basis of fractals is the idea of self-similarity, and a widely used tool in the composition and the analysis of music is the idea of patterns and repetition.

Fractal Dimension of Music

Is Music Fractal?

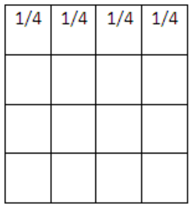
We know that if we generate a melody through an integer sequence with known fractal qualities, we will come up with music that has those innately fractal qualities. The question still remains: does music naturally have fractal qualities? Research has found that many things in nature do. Mandelbrot considers this when he looks at the length of the coastline of Britain. At first, it seems as if measuring the length would be as simple as measuring any line. It turns out this is not the case. Since any coastline is very jagged, the smaller the measuring stick used the longer the distance calculated, until at some point we consider the length infinite. We need some other tool when thinking about this jagged line. This tool happens to be fractal geometry [**2**].

Fractal Dimension



When we look at the Koch Snowflake, we run into the same problem. Since there are an infinite number of iterations, the length of the border of the snowflake will grow infinitely. Even though it looks like a closed shape with a finite boundary, there is actually an infinite boundary. Since the snowflake is a fractal, it is not one-dimensional or two-dimensional since those do not exactly make sense in this geometry. Instead, we consider its fractal dimension [**3**].

When thinking about measuring the dimension of a fractal, we must first think about how to measure the dimension of any shape. If we consider a unit square broken



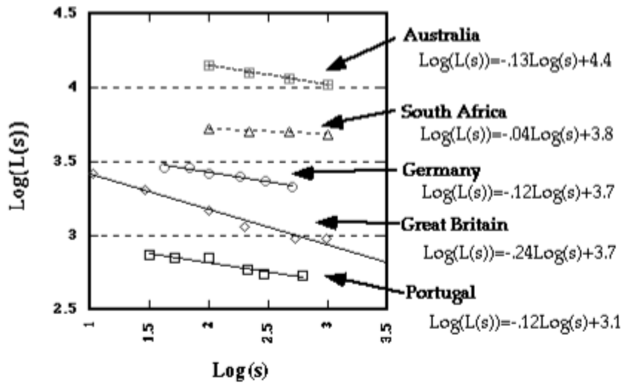
up into sixteen parts, then the area of the square is $A = 16(1/4)^2 = 1$. If we let *r* be the ratio of a small square within the square to the whole square, *n* be the number of small squares that it is broken into, and *d* be the dimension of the square, then $1 = n(r)^d$. Solving for *d*, we can use this formula to find the dimension of any object. Solving for *d* gives us

$$d = \frac{\log n}{\log(1/r)}$$

Generalizing *n* and *r*, we can say that *n* is the number of pieces used to construct the whole piece, and $S = 1/r$ is the scaling of the pieces [**4**]. Returning to our snowflake example, in constructing the snowflake we break each side into 3 pieces, and then use 4 pieces to reconstruct the side. Therefore, the fractal dimension of the snowflake is

$$d = \frac{\log 4}{\log 3} \approx 1.26$$

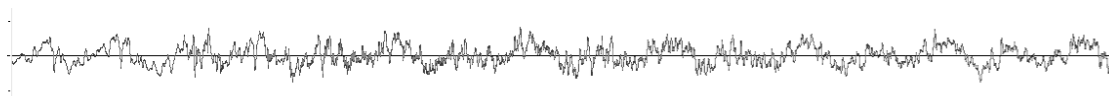
The coastline of Britain, however, is not an iterative line. We must use some other idea to find its fractal dimension. This time, we will use *ε* as the length of the measuring stick, and *L*(*ε*) as the estimated length using that measuring stick. If we decrease *ε*, then *L*(*ε*) will increase [**2**]. After taking a variety of measurements using different values for *ε*, we plot log(*ε*) on the x-axis, and log(*L*(*ε*)) on the y-axis. Using the least-squares method, we come up with a linear line with a negative slope. The slope is defined as (1-*d*). A study was done on a variety of slopes using *s* instead of *ε*, and the plot is the following graph [**14**]:



Since the coast of South Africa is of dimension 1.04, this means that it is very close to being a smooth line, and thus it is not very rough. This also shows that Great Britain has the roughest coast, with a fractal dimension of 1.24. To show that this is true, compare the coastlines in the following maps:

Fractal Music

In finding the dimension of these jagged lines, we know that in actuality, the coasts are going between the dimensions of length and width. Music also has dimension. Instead of length and width, it has the dimensions of time and amplitude. When looking at fractal dimension, we are no longer doing a melodic study as we have been previously doing. Instead, we are doing an amplitude analysis. When music is mapped with time on the x-axis and amplitude on the y-axis, we are given a jagged line much like the edge of a coast. Thus, in the same way we can find the fractal dimension of a coast, we can also find the fractal dimension of music. Here is an example of about one second of “Come Together” by The Beatles mapped out using Audacity:



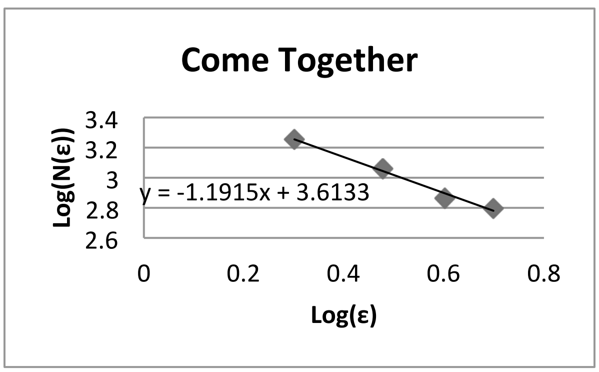
In an undergraduate study, Perrin Meyer observes the fractal dimension to music that is mapped out, and sought to find out what that dimension is for a variety of styles of music. He used the box-counting algorithm,

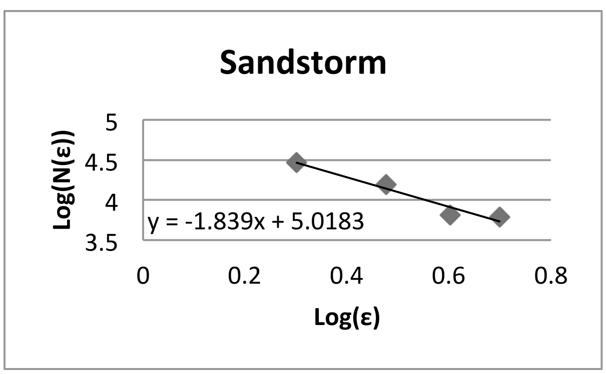
$$\log(N(\varepsilon)) = -D * \log(\varepsilon)$$

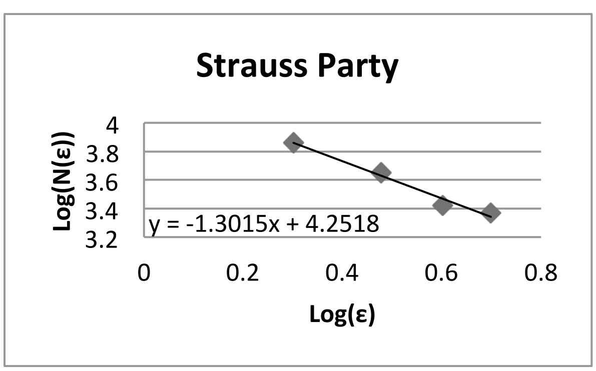
where ε is the length of the measuring stick used and $N(\varepsilon)$ is the number of measuring sticks used. After plotting $\log(N(\varepsilon))$ versus $\log(\varepsilon)$, D is the fractal dimension. As a result of this study, he concludes that music averages a fractal dimension of 1.65, varying from 1.6 to 1.69 regardless of the genre of music [**15**].

To test his conclusion, we will set up our own test using three different styles of music. We will use a rock song, “Come Together” by The Beatles, a techno song, “Sandstorm” by Darude, and a classical song, “Strauss Party” by André Rieu. To limit our data set, we will only use portions of these songs no longer than ten seconds. If we pair up each data sets by 2s, 3s, 4s, and 5s, then, using Maple, we can find the distance traveled between each set. In essence, we are using “measuring sticks” of sizes 2, 3, 4, and 5.

The resulting graphs of plotting log(N(ε)) versus log(ε) can be found below. The results are that "Come Together" is of fractal dimension 1.1915, "Sandstorm" is 1.839, and "Strauss Party" is 1.3015. The closer to 2, the rougher the line, and the closer to 1, the smoother the line. It makes sense that the techno song is closer to 2 than the others, since it is "noiser." While certainly more tests should be done to see if there is a distinct difference in fractal dimension in various types of music, it can be seen with these that there is more of a deviation from 1.65 than Meyer supposed.







Conclusion

The aesthetic quality of the music we created, as with any type of music, is subjective. To the mathematician who created it, it may be surprisingly interesting to listen to as one may hear patterns and repetition, as with the Morse-Thue Sequence. To a musician, it may sound not much more than mediocre. To neither, it may sound dull and uninspired.

Gustavo Díaz-Jerez created a fractal music software called FractMus [**16**]. It uses sequences created by choatic systems to create music. The user can choose a variety of combinations of different algorithms and instruments. The problem is not everything one chooses will sound very pleasing. In fact, most of the combinations sound random. However, if one works at it, one can create songs that have depth and interest. In the same way, fractal art pieces are only eye catching and interesting when a person is able to manipulate it.

When considering the usefulness of fractal music, Michael Peters says that "it might prove more rewarding to embrace the limits of fractal music, emphasizing its nonhuman aesthetics and acknowledging its unpredictability, and to combine it with improvisation" [**17**]. Music is meant as something that is an extention of ourselves to convey something to the listener. It is purpose-driven, whether that purpose is to inform us, envoke some sort of emotional response, or allow us to escape. Music that is created by a computer, with little human intervention, misses a part of that. It is limiting. While fractals can help us to understand the structure of music or provide us with a tool to delve into musical composition, they can never replace human musical composition.

Our discussion also leaves us with more questions that have yet to be investigated. Does the fractal dimension of music that is mathematically generated by fractals have a connection to the fractal dimension of the shape or structure it comes from? Is there a way to smoothly transition from a fractal artpiece to fractal music? In what other ways are there natural connections between fractals and music? These topics that we have discussed form a good starting point for future research.

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